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# Integrable discretizations of chiral models via deformation of the differential calculus

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**Abstract.** A construction of conservation laws for chiral models (generalized  $\sigma$ -models) on a two-dimensional spacetime continuum using differential forms is extended in such a way that it also comprises corresponding discrete versions. This is achieved via a deformation of the ordinary differential calculus. In particular, the nonlinear Toda lattice results in this way from the linear (continuum) wave equation. The method is applied to several further examples. We also construct Lax pairs and Bäcklund transformations for the class of models considered in this work.

## 1. Introduction

Some years ago we observed that a certain deformation of the ordinary calculus of differential forms on  $\mathbb{R}^n$  can be used to discretize classical continuum field theories [1]. For this purpose one has to formulate the theory in terms of differential forms. The deformation of the differential calculus then induces a corresponding deformation of the theory built on it. In particular, the Wilson loop formulation of lattice gauge theory originates in this way from continuum Yang–Mills theory. In this paper we present another application of the method, a discretization of chiral models<sup>†</sup> (or generalized  $\sigma$ -models) preserving complete integrability. After a brief introduction to deformations of the ordinary differential calculus on  $\mathbb{R}^2$  we generalize the derivation of conservation laws given for chiral models in [3]. As an example, we then derive the nonlinear Toda lattice [4] from the continuum wave equation. Further examples illustrate the method and reveal its present limitations. We also construct Lax pairs for the discretized chiral models and present Bäcklund transformations.

## 2. Deformation of the ordinary differential calculus on $\mathbb{R}^2$

In the ordinary differential calculus on manifolds, functions commute with differentials. It is possible, however, to dispense with this property while keeping the familiar rules for the exterior derivative (see [1] for details). The latter are consistent with the following commutation relations:

$$dt f(t, x) = f(t + \tau, x) dt \quad dx f(t, x) = f(t, x + \lambda) dx \quad (1)$$

where  $\tau$  and  $\lambda$  are real parameters,  $f$  is a function on  $\mathbb{R}^2$  and  $t, x$  are the canonical coordinate functions on  $\mathbb{R}^2$ . In particular,

$$[dt, t] = \tau dt \quad [dt, x] = [dx, t] = 0 \quad [dx, x] = \lambda dx. \quad (2)$$

<sup>†</sup> See [2] and references therein.

Using these relations we obtain for  $\tau, \lambda \neq 0$

$$df = (\partial_{+t} f) dt + (\partial_{+x} f) dx = dt (\partial_{-t} f) + dx (\partial_{-x} f) \quad (3)$$

with

$$\partial_{+x} f(t, x) = \frac{1}{\lambda} [f(t, x + \lambda) - f(t, x)] \quad \partial_{-x} f(t, x) = \frac{1}{\lambda} [f(t, x) - f(t, x - \lambda)] \quad (4)$$

and similar expressions for  $\partial_{\pm t} f(t, x)$ . This shows that for  $\tau, \lambda \neq 0$  the differential calculus is actually defined on the algebra  $\mathcal{A}$  of *all* real functions on  $\mathbb{R}^2$ <sup>†</sup>. The deformed differential calculus has more ‘constants’ than the ordinary one. From the above formulae one finds that  $df = 0$  if and only if  $f(t + \tau, x) = f(t, x)$  and  $f(t, x + \lambda) = f(t, x)$ . Let us choose a point, say 0, in  $\mathbb{R}$  and construct the ideal  $\mathcal{I}_\tau$  of  $\mathcal{A}$  generated by  $f_\tau(t, x) - f_\tau(0, x)$  for all functions  $f_\tau \in \mathcal{A}$  which are periodic (with period  $\tau$ ) in the first argument. Then the algebra  $\mathcal{A}/\mathcal{I}_\tau$  is isomorphic to the algebra of real-valued functions on  $\mathbb{Z} \times \mathbb{R}$ . If  $\mathcal{I}_{\tau, \lambda}$  denotes the ideal of functions generated by  $f_{\tau, \lambda}(t, x) - f_{\tau, \lambda}(0, 0)$  for all functions periodic in both arguments, one finds that  $\mathcal{A}/\mathcal{I}_{\tau, \lambda}$  is isomorphic to the algebra of real-valued functions on  $\mathbb{Z}^2$ .

In what follows, depending on whether the parameters  $\tau$  and  $\lambda$  are zero or not,  $M$  denotes either  $\mathbb{R}^2$ ,  $\mathbb{R} \times \mathbb{Z}$ ,  $\mathbb{Z} \times \mathbb{R}$ , or  $\mathbb{Z}^2$ . Correspondingly, let  $\mathcal{A}$  denote the algebra of real functions  $f(t, x)$  which are smooth in both arguments, or functions  $f_k(t) := f(t, k\lambda)$  smooth in the first argument, respectively functions  $f_k(x) := f(k\tau, x)$  smooth in the second argument, or the algebra of all real functions  $f_k(n) := f(n\tau, k\lambda)$  on  $\mathbb{Z}^2$ <sup>‡</sup>. The differential calculus defined above is then a differential calculus on the algebra  $\mathcal{A}$ . Acting with the exterior derivative on (2) we obtain the 2-form relations

$$dx dx = 0 \quad dt dx + dx dt = 0 \quad dt dt = 0. \quad (5)$$

Hence  $dt dx$  is a basis of the space of 2-forms  $\Omega^2$  as a left or right  $\mathcal{A}$ -module. There are no forms of higher grade, i.e.  $\Omega^r = \{0\}$  for  $r \geq 3$ . Let  $\Omega := \bigoplus_{r \geq 0} \Omega^r$  denote the differential algebra (where  $\Omega^0 = \mathcal{A}$ ).

We introduce an inner product  $(, ) : \Omega \times \Omega \rightarrow \mathcal{A}$  via  $(f, g) = f g$  for  $f, g \in \mathcal{A}$ ,

$$(dt, dt) = -1 \quad (dx, dx) = 1 \quad (dt, dx) = 0 \quad (dt dx, dt dx) = -1 \quad (6)$$

and

$$(\psi, \phi) = (\phi, \psi) \quad (\psi, f \phi) = f (\psi, \phi) \quad (7)$$

for  $\psi, \phi \in \Omega$ .<sup>§</sup> If  $\psi$  and  $\phi$  have different grades, then  $(\psi, \phi)$  is set to zero. As a consequence of these definitions we find

$$(\psi f, \phi) = (\psi, \phi f). \quad (8)$$

A Hodge  $*$ -operator can now be introduced as an  $\mathbb{R}$ -linear operator on  $\Omega$  via

$$(dt dx, \phi * \psi) := -(\phi, \psi). \quad (9)$$

It satisfies the relations

$$*(\psi f) = f * \psi \quad \psi * \phi = \phi * \psi \quad (10)$$

<sup>†</sup> In order to perform the limit  $\tau \rightarrow 0$  ( $\lambda \rightarrow 0$ ) we have to restrict  $\mathcal{A}$  to those functions which are differentiable in  $t(x)$ .

<sup>‡</sup> Here  $k$  and also  $n$  should be regarded as the canonical coordinate function on  $\mathbb{Z}$ .

<sup>§</sup> Our inner product corresponds to a metric with Lorentzian signature. The formalism works as well with a Euclidean metric.

where  $\phi$  and  $\psi$  must have the same grade, and

$$*1 = dt dx \quad * dt = -dx \quad * dx = -dt \quad *(dt dx) = -1. \tag{11}$$

Furthermore

$$**\psi(t, x) = (-1)^{r+1} \psi(t - \tau, x - \lambda) \tag{12}$$

for  $\psi \in \Omega^r$ .

The notion of an integral generalizes to our deformed differential calculus in a natural way [1]. In the following sections we only need to consider one-dimensional integrals. It is therefore sufficient here to define the integral for functions on the lattice  $\mathbb{Z}$ . An indefinite integral is indeed determined by

$$d \int^x f(x') dx' = f(x) dx \tag{13}$$

up to a constant. For a corresponding definite integral over a lattice interval (i.e. an integer multiple of  $\lambda$ ), one obtains

$$\int_{m\lambda}^{n\lambda} f(x) dx = \lambda \sum_{k=m}^{n-1} f(k\lambda) \tag{14}$$

where  $m, n \in \mathbb{Z}, n > m$  (see [1] for details).

### 3. Chiral models and conservation laws in two dimensions

In this section we essentially follow Brezin *et al* [3]. However, the following not only works for the ordinary differential calculus but also for its deformations considered in the previous section.  $\mathcal{B}$  denotes a finite dimensional algebra of matrices and  $\mathcal{B}^*$  the group of invertible elements of  $\mathcal{B}$ . Let  $g : M \rightarrow \mathcal{B}$  be invertible, i.e.  $g(t, x) \in \mathcal{B}^*$  for all  $t, x$ . In terms of

$$A := g^{-1} dg \tag{15}$$

the field equations of a chiral model (generalized  $\sigma$ -model) are

$$d * A = 0. \tag{16}$$

An infinite set of conservation laws of such a model is obtained as follows. Let  $\Gamma$  be the space of fields  $\Psi : M \rightarrow \mathcal{B}$  and  $D : \Gamma \rightarrow \Omega^1 \otimes_{\mathcal{A}} \Gamma$  the exterior covariant derivative given by

$$D\Psi = d\Psi + A\Psi. \tag{17}$$

Since  $A$  is a ‘pure gauge’ we have

$$F := dA + AA = 0. \tag{18}$$

Moreover, using equations (16) and (10) we find

$$d * (A^i_j \Psi^j_k) = d(\Psi^j_k * A^i_j) = (d\Psi^j_k) * A^i_j = A^i_j * d\Psi^j_k \tag{19}$$

and thus

$$d * D\Psi = D * d\Psi. \tag{20}$$

Let  $J^{(m)} : M \rightarrow \Omega^1 \otimes \mathcal{B}$  be a conserved current, i.e.

$$d * J^{(m)} = 0. \tag{21}$$

Since the first cohomology group of  $M$  is trivial, i.e.  $H^1(M) = \{0\}$ , there exists  $\chi^{(m)} : M \rightarrow \mathcal{B}$  such that

$$J^{(m)} = *d\chi^{(m)}. \tag{22}$$

We define

$$J^{(m+1)} := D\chi^{(m)}. \tag{23}$$

Starting with  $\chi^{(0)} = I$ , the identity matrix, we have  $J^{(0)} = 0$ . Then  $J^{(1)}$  is conserved as a consequence of the field equation (16). Using

$$d * J^{(m+1)} = d * D\chi^{(m)} = D * d\chi^{(m)} = DJ^{(m)} = DD\chi^{(m-1)} = F \chi^{(m-1)} = 0 \tag{24}$$

it follows by induction that  $J^{(m)}$ ,  $m > 1$ , is also conserved. We therefore obtain an infinite number of conserved charges given by

$$Q^{(m)} := \int_{t=\text{constant}} *J^{(m)} \tag{25}$$

(provided that the currents satisfy suitable fall off conditions at spatial infinity). Let us calculate the first two of them. We have  $J^{(1)} = DI = A$  and therefore

$$Q^{(1)} = \int_{t=\text{constant}} *A = - \int A_0(t - \tau, x) dx \tag{26}$$

where  $A = A_0 dt + A_1 dx$ . Since  $*d\chi^{(1)} = J^{(1)} = A$  we find  $d\chi^{(1)} = *A(t + \tau, x + \lambda)$  by use of (12). Hence

$$\chi^{(1)}(t, x) = - \int^x A_0(t, x' + \lambda) dx'. \tag{27}$$

and

$$*J^{(2)} = *D\chi^{(1)} = J^{(1)} + *(A \chi^{(1)}). \tag{28}$$

Using equation (10) we find

$$Q^{(2)} = \int_{t=\text{constant}} *J^{(2)} = \int A_1(t, x) dx - \int A_0(t - \tau, x) \chi^{(1)}(t, x) dx. \tag{29}$$

Let us introduce

$$\chi := \sum_{m=0}^{\infty} \gamma^m \chi^{(m)} \tag{30}$$

where  $\gamma$  is a parameter. From equations (22) and (23) we obtain

$$*d\chi^{(m+1)} = D\chi^{(m)}. \tag{31}$$

Multiplying by  $\gamma^{m+1}$  and summing over  $m$  leads to

$$*d\chi = \gamma D\chi. \tag{32}$$

The field equations (16) are integrability conditions of the linear system (32). This is seen as follows. Acting with  $D$  on (32) and using  $F = 0$ , we find  $D * d\chi = 0$ . Applying  $*$  to (32) we get  $d\chi(t - \tau, x - \lambda) = \gamma * D\chi$  and thus  $d * D\chi = 0$  which, together with  $D * d\chi = 0$ , implies  $d * A = 0$ . Introducing  $J := *d\chi$  we have

$$Q(t) := \int_{t=\text{constant}} *J = \int_{t=\text{constant}} d\chi(t - \tau, x - \lambda) = \chi(t - \tau, x)|_{-\infty}^{+\infty}. \tag{33}$$

### 4. Examples

#### 1. Toda lattice

Let  $\mathcal{B}$  be the algebra  $\mathbb{R}$  of real numbers and let us write  $g = e^{-q}$  with a function  $q : M \rightarrow \mathbb{R}$ . The field equations (16) then read

$$d * (e^q de^{-q}) = 0. \tag{34}$$

(a)  $\tau = \lambda = 0$ . Then equation (34) is just the wave equation  $d * dq = 0$  and

$$\partial_t^2 q - \partial_x^2 q = 0. \tag{35}$$

(b)  $\tau = 0$  and  $\lambda \neq 0$ . Then

$$A = -\dot{q}_k dt + \frac{1}{\lambda} (e^{q_k - q_{k+1}} - 1) dx \quad *A = \dot{q}_k dx - \frac{1}{\lambda} (e^{q_{k-1} - q_k} - 1) dt \tag{36}$$

where  $\dot{q}_k := dq_k/dt$ . The field equations become

$$\ddot{q}_k = \frac{1}{\lambda^2} [e^{q_{k-1} - q_k} - e^{q_k - q_{k+1}}] \tag{37}$$

which are those of the nonlinear Toda lattice [4]. The conserved charges can be obtained using the method described in the previous section. In particular, evaluation of (26) using (14) leads to

$$Q^{(1)} = \lambda \sum_{k=-\infty}^{\infty} \dot{q}_k \tag{38}$$

which is the total momentum. Furthermore

$$\chi^{(1)}(t, k\lambda) = \lambda \sum_{\ell=-\infty}^k \dot{q}_\ell(t) \tag{39}$$

(modulo addition of a constant) and thus

$$Q^{(2)} = \sum_{k=-\infty}^{\infty} \left[ e^{q_k - q_{k+1}} - 1 + \lambda^2 \sum_{\ell \leq k} \dot{q}_k \dot{q}_\ell \right]. \tag{40}$$

This yields

$$Q^{(2)} - \frac{1}{2} [Q^{(1)}]^2 = \sum_k \left[ \frac{1}{2} \lambda^2 \dot{q}_k^2 + e^{q_k - q_{k+1}} - 1 \right] \tag{41}$$

which is the total energy.

(c)  $\tau \neq 0, \lambda \neq 0$ . Then

$$A = \frac{1}{\tau} (e^{q_k^{(n)} - q_k^{(n+1)}} - 1) dt + \frac{1}{\lambda} (e^{q_k^{(n)} - q_{k+1}^{(n)}} - 1) dx \tag{42}$$

$$*A = -\frac{1}{\tau} (e^{q_k^{(n-1)} - q_k^{(n)}} - 1) dx - \frac{1}{\lambda} (e^{q_{k-1}^{(n)} - q_k^{(n)}} - 1) dt \tag{43}$$

and the field equations are

$$\frac{1}{\tau^2} [e^{q_k^{(n-1)} - q_k^{(n)}} - e^{q_k^{(n)} - q_k^{(n+1)}}] = \frac{1}{\lambda^2} [e^{q_{k-1}^{(n)} - q_k^{(n)}} - e^{q_k^{(n)} - q_{k+1}^{(n)}}]. \tag{44}$$

This describes a discrete-time Toda lattice<sup>†</sup>. Written in the form

$$e^{q_k(n)-q_k(n+1)} = e^{q_k(n-1)-q_k(n)} - \frac{1}{c^2} [e^{q_{k-1}(n)-q_k(n)} - e^{q_k(n)-q_{k+1}(n)}] \quad (45)$$

with  $c := \lambda/\tau$ , we see that the right-hand side is not necessarily positive in contrast to the left-hand side. As a consequence, there is a constraint on initial values. Exact solutions are given by

$$q_k(n) = f(k+n) - 2k \ln c \quad (46)$$

where  $f$  is an arbitrary function. These ‘left movers’ do not have a right-moving counterpart. From equations (42) and (23) it is evident that  $q_k(n)$  enters the conserved charges only through the quantities

$$U_k(n) := e^{q_k(n)-q_{k+1}(n)} \quad V_k(n) := e^{q_k(n)-q_k(n+1)} \quad (47)$$

for which we obtain the following first-order system:

$$U_k(n+1) = \frac{V_{k+1}(n)}{V_k(n)} U_k(n) \quad (48)$$

$$V_k(n+1) = V_k(n) + c^{-2} [U_k(n+1) - U_{k-1}(n+1)].$$

The first conserved charge is

$$Q^{(1)}(n) = c \sum_k [1 - V_k(n)]. \quad (49)$$

When  $c \neq 1$ , the solutions (46) become infinite either for  $k \rightarrow \pm\infty$  or for  $n \rightarrow \pm\infty$ . The quantities  $U_k(n)$ ,  $V_k(n)$  and thus also the conserved charges may remain finite, however.

## 2. $GL(n, \mathbb{R})$ -models

We express an element  $g \in GL(n, \mathbb{R})$  as  $g = \pm e^{-q} s$  with real  $q$  and  $s \in SL(n, \mathbb{R})$ . In the following we only discuss the case  $\tau = 0$  and  $\lambda \neq 0$ . The field equations (16) then split into the two parts

$$\ddot{q}_k = \frac{1}{n\lambda^2} [e^{q_{k-1}-q_k} \operatorname{tr}(s_{k-1}^{-1} s_k) - e^{q_k-q_{k+1}} \operatorname{tr}(s_k^{-1} s_{k+1})] \quad (50)$$

and

$$(s_k^{-1} \dot{s}_k)' = \frac{1}{\lambda^2} \left( e^{q_k-q_{k+1}} \left[ s_k^{-1} s_{k+1} - \frac{I}{n} \operatorname{tr}(s_k^{-1} s_{k+1}) \right] - e^{q_{k-1}-q_k} \left[ s_{k-1}^{-1} s_k - \frac{I}{n} \operatorname{tr}(s_{k-1}^{-1} s_k) \right] \right) \quad (51)$$

(where  $I$  is the identity matrix). The first equation resembles that of the nonlinear Toda lattice to which it reduces for  $s$  not depending on  $k$ .

## 3. A $GL(1, \mathbb{C})$ -model

We write an element of  $GL(1, \mathbb{C})$  in the form  $g = e^{-q} e^{i\theta}$  with real  $q$  and  $\theta$ . In this case the field equations for  $\tau = 0$  and  $\lambda \neq 0$  read

$$\ddot{q}_k = \frac{1}{\lambda^2} [e^{q_{k-1}-q_k} \cos(\theta_k - \theta_{k-1}) - e^{q_k-q_{k+1}} \cos(\theta_{k+1} - \theta_k)] \quad (52)$$

$$\ddot{\theta}_k = \frac{1}{\lambda^2} [e^{q_{k-1}-q_k} \sin(\theta_k - \theta_{k-1}) - e^{q_k-q_{k+1}} \sin(\theta_{k+1} - \theta_k)]. \quad (53)$$

<sup>†</sup> See also [5] and references therein for a class of discrete time generalized Toda lattices.

A reduction to a  $U(1)$ -model by setting  $q = 0$  leads to a constraint. On the other hand, setting  $\theta = 0$  simply leads us back to the nonlinear Toda lattice.

#### 4. A $GL(1, \mathbb{H})$ -model

Here  $\mathbb{H}$  denotes the quaternions. Again, we write an element in the form  $g = e^{-q} u$  with real  $q$  and  $u = \alpha + \beta i + \gamma j + \delta k$ ,  $u\bar{u} = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$ . For  $\tau = 0$  and  $\lambda \neq 0$  the field equations (16) take the form

$$\ddot{q}_k = \frac{1}{2\lambda^2} \left[ e^{q_{k-1}-q_k} (\bar{u}_{k-1}u_k + \bar{u}_k u_{k-1}) - e^{q_k-q_{k+1}} (\bar{u}_k u_{k+1} + \bar{u}_{k+1}u_k) \right] \tag{54}$$

$$(\bar{u}_k \dot{u}_k)' = \frac{1}{2\lambda^2} \left[ e^{q_{k-1}-q_k} (\bar{u}_{k-1}u_k - \bar{u}_k u_{k-1}) - e^{q_k-q_{k+1}} (\bar{u}_k u_{k+1} - \bar{u}_{k+1}u_k) \right]. \tag{55}$$

Although  $GL(1, \mathbb{H})$  is isomorphic to  $GL(1, \mathbb{R}) \times U(2)$ , if we had chosen the latter representation, some unpleasant constraints would have shown up.

#### 5. The $O(n)$ $\sigma$ -model

Let  $S \in \mathbb{R}^n$  be a unit vector,  $S \bullet S = 1$ . The matrix  $g = I - 2P$  with  $P = S \otimes S$  is orthogonal. We have  $P^2 = P$  and therefore  $g^{-1} = g$ . Hence

$$A = 2(P dP - dP P) \tag{56}$$

using  $dP = dP^2 = P dP + dP P$ . In terms of  $S$  we have

$$A = 2[S \otimes dS - dS \otimes S + 2S \otimes (S \bullet dS)S] \tag{57}$$

where we used  $dS \bullet S + S \bullet dS = 0$ † For  $\tau \neq 0$  and  $\lambda \neq 0$ , we obtain

$$\begin{aligned} *A = & -2[S^{-x} \otimes \partial_{-x}S - \partial_{-x}S \otimes S + 2(S^{-x} \bullet \partial_{-x}S)S^{-x} \otimes S] dt \\ & -2[S^{-t} \otimes \partial_{-t}S - \partial_{-t}S \otimes S + 2(S^{-t} \bullet \partial_{-t}S)S^{-t} \otimes S] dx \end{aligned} \tag{58}$$

with  $S^{-t}(t, x) := S(t - \tau, x)$  and a corresponding definition for  $S^{-x}$ . The field equations  $d * A = 0$  now take the form

$$\begin{aligned} \partial_{+t} [S^{-t} \otimes \partial_{-t}S - \partial_{-t}S \otimes S + 2(S^{-t} \bullet \partial_{-t}S)S^{-t} \otimes S] \\ - \partial_{+x} [S^{-x} \otimes \partial_{-x}S - \partial_{-x}S \otimes S + 2(S^{-x} \bullet \partial_{-x}S)S^{-x} \otimes S] = 0. \end{aligned} \tag{59}$$

In the limit  $\tau, \lambda \rightarrow 0$  we get

$$S \otimes (\partial_t^2 - \partial_x^2)S = (\partial_t^2 - \partial_x^2)S \otimes S. \tag{60}$$

Acting on this equation from the left with  $S \bullet$  and using  $S \bullet dS = 0$  (which holds in the case  $\tau = \lambda = 0$ ), we recover the field equations of the classical nonlinear  $\sigma$ -model on  $\mathbb{R}^2$

$$\partial_t^2 S - \partial_x^2 S + (\partial_t S \bullet \partial_t S)S - (\partial_x S \bullet \partial_x S)S = 0. \tag{61}$$

The conserved charges of this model were first obtained in [7] by means of inverse scattering methods. When  $\tau = 0$  and  $\lambda \neq 0$ , the field equations are

$$\begin{aligned} S_k \otimes \partial_t^2 S_k - \partial_t^2 S_k \otimes S_k = \frac{1}{\lambda^2} [2(S_k \bullet S_{k+1})S_k \otimes S_{k+1} - 2(S_{k-1} \bullet S_k)S_{k-1} \otimes S_k \\ - S_{k+1} \otimes S_{k+1} + S_{k-1} \otimes S_{k-1}] \end{aligned} \tag{62}$$

† Note that (for  $\tau \neq 0$  or  $\lambda \neq 0$ ) the non-commutativity between functions and differentials leads to  $S \bullet dS \neq 0$ , in general.



in terms of  $S_k(t) := S(t, \lambda k)$ . Acting from the left $\ddagger$  with  $S_k \bullet$  and using  $S_k \bullet S_k = 1$  (which implies  $S_k \bullet \partial_t^2 S_k = -\partial_t S_k \bullet \partial_t S_k$ ) we obtain

$$\ddot{S}_k + (\dot{S}_k \bullet \dot{S}_k) S_k + \frac{1}{\lambda^2} [2(S_k \bullet S_{k-1})^2 S_k - (S_k \bullet S_{k+1}) S_{k+1} - (S_k \bullet S_{k-1}) S_{k-1}] = 0. \quad (63)$$

This does not exhaust equations (62), however. In addition we have the constraint

$$S_{k+1} \otimes S_{k+1} - S_{k-1} \otimes S_{k-1} + (S_{k-1} \bullet S_k) S_k \otimes S_{k-1} - (S_k \bullet S_{k+1}) S_k \otimes S_{k+1} \\ + (S_{k-1} \bullet S_k) S_{k-1} \otimes S_k - (S_k \bullet S_{k+1}) S_{k+1} \otimes S_k = 0. \quad (64)$$

The problem is that this constraint is not automatically respected by (63) so that differentiation with respect to time generates additional equations of motion. The conclusion is that the equations governing our lattice  $O(n)$   $\sigma$ -model are not ‘good’ equations. There are at least simple exact solutions like those given by

$$S_k \bullet S_{k+1} = 0 \quad S_{k+2} = \pm S_k \quad \ddot{S}_k + (\dot{S}_k \bullet \dot{S}_k) S_k = 0. \quad (65)$$

The appearance of a constraint, as in our last example, is a rather general feature which can be understood as follows. The equation  $d*(g^{-1}dg) = 0$ , where  $g$  has values in a group, takes the following form for  $\tau = 0$  and  $\lambda \neq 0$

$$\partial_t(g_k^{-1} \partial_t g_k) = \frac{1}{\lambda^2} (g_k^{-1} g_{k+1} - g_{k-1}^{-1} g_k). \quad (66)$$

Whereas the left-hand side is in the Lie algebra of the group, the right-hand side lives in the group algebra. The deviation of the group algebra from the Lie algebra results in constraints. In case of the orthogonal group  $O(n)$ , the left-hand side of the last equation is an antisymmetric matrix, but this property is not shared by the right-hand side. The symmetric part of the matrix equation is then a constraint. This problem does not appear for  $GL(n)$ -models. In case of the  $O(n)$ -model, one may take into consideration a simultaneous deformation of  $S \bullet S = 1$  as an attempt to get rid of the constraints.

## 5. Lax pairs for the chiral models

In this section we construct Lax pairs for our chiral models. This establishes contact with other formulations of integrable models (see [6], for example). The starting point is the equation  $*d\chi = \gamma D\chi$  derived in section 3.

1.  $M = \mathbb{R}^2$

Evaluation of equation (32) leads to

$$\partial_t \chi = -\gamma [\partial_x \chi + (g^{-1} \partial_x g) \chi] \quad \partial_x \chi = -\gamma [\partial_t \chi + (g^{-1} \partial_t g) \chi]. \quad (67)$$

Solving this system for the partial derivatives of  $\chi$ , we obtain

$$\partial_x \chi = L \chi \quad \partial_t \chi = M \chi \quad (68)$$

$\ddagger$  Acting from the right with  $\bullet S_k$  yields the same equation but with  $S_{k+1}$  replaced by  $S_{k-1}$  in the quadratic term on the right-hand side.

with

$$L(t, x; \gamma) = \frac{\gamma}{1 - \gamma^2} g^{-1} (-\partial_x g + \gamma \partial_t g) \quad M(t, x; \gamma) = \frac{\gamma}{1 - \gamma^2} g^{-1} (\gamma \partial_x g - \partial_t g). \tag{69}$$

In terms of  $L$  and  $M$  the integrability conditions for the system (68), which are the field equations, read

$$\partial_t L - \partial_x M + [L, M] = 0. \tag{70}$$

2.  $M = \mathbb{R} \times \mathbb{Z}$

In this case equation (32) yields

$$\dot{\chi}_k = \frac{\gamma}{\lambda} [g_k^{-1} g_{k+1} \chi_{k+1} - \chi_k] \quad \chi_k - \chi_{k-1} = -\gamma \lambda [\dot{\chi}_k + g_k^{-1} \dot{g}_k \chi_k]. \tag{71}$$

Introducing  $\psi_k := \chi_{k-1}$ ,  $\phi_k := \gamma g_k \chi_k$  and  $\xi_k := (\phi_k, \psi_k)^T$ , we obtain

$$\xi_{k+1} = L_k \xi_k \quad \dot{\xi}_k = M_k \xi_k \tag{72}$$

with

$$L_k(t; \gamma) = \gamma^{-1} \begin{pmatrix} (\gamma^{-1} + \gamma) I + \lambda \dot{g}_k g_k^{-1} & -g_k \\ g_k^{-1} & 0 \end{pmatrix} \tag{73}$$

and

$$M_k(t; \gamma) = \lambda^{-1} \begin{pmatrix} -\gamma^{-1} I & g_k \\ -g_{k-1}^{-1} & \gamma I \end{pmatrix}. \tag{74}$$

The field equations of the chiral model are now obtained as integrability conditions of the linear system (72) in the form

$$\partial_t L + L_k M_k - M_{k+1} L_k = 0. \tag{75}$$

3.  $M = \mathbb{Z}^2$

Now equation (32) leads to

$$\begin{aligned} \chi_k(n) - \chi_k(n-1) &= -\frac{\gamma}{c} [g_k(n)^{-1} g_{k+1}(n) \chi_{k+1}(n) - \chi_k(n)] \\ \chi_k(n) - \chi_{k-1}(n) &= -\gamma c [g_k(n)^{-1} g_k(n+1) \chi_k(n+1) - \chi_k(n)] \end{aligned} \tag{76}$$

where  $c := \lambda/\tau$ . Let us introduce  $\psi_k(n) := \chi_{k-1}(n-1)$ ,  $\phi_k(n) := \gamma g_k(n) \chi_k(n)$  and  $\xi_k(n) := (\phi_k(n), \psi_k(n))^T$ . Then

$$\xi_{k+1}(n) = L_k(n) \xi_k(n) \quad \xi_k(n+1) = M_k(n) \xi_k(n) \tag{77}$$

with

$$L_k(n; \gamma) = (1 - \gamma c)^{-1} \hat{L}_k(n; \beta) \quad M_k(n; \gamma) = (c - \gamma)^{-1} c^{-1} \hat{M}_k(n; \beta) \tag{78}$$

where

$$\begin{aligned} \hat{L}_k(n; \beta) &= \begin{pmatrix} \beta I - c^2 g_k(n) [g_k(n-1)]^{-1} & c g_k(n) \\ -c [g_k(n-1)]^{-1} & I \end{pmatrix} \\ \hat{M}_k(n; \beta) &= \begin{pmatrix} \beta I - g_k(n) [g_{k-1}(n)]^{-1} & c g_k(n) \\ -c [g_{k-1}(n)]^{-1} & c^2 I \end{pmatrix} \end{aligned} \tag{79}$$

and  $\beta = 1 - c(\gamma + \gamma^{-1}) + c^2$ . Now the field equations are obtained as

$$\hat{L}_k(n+1)\hat{M}_k(n) = \hat{M}_{k+1}(n)\hat{L}_k(n) \quad (80)$$

(suppressing the argument  $\beta$ ). The matrices  $\hat{L}_k$  and  $\hat{M}_k$  are invertible if  $\beta \neq 0$ .

## 6. Bäcklund transformations for the chiral models

If  $g$  is an exact solution of the field equation for a chiral model, then also  $h$  if

$$h^{-1}dh - g^{-1}dg = *dF \quad (81)$$

with a  $\mathcal{B}$ -valued function  $F$ . A suitable choice for  $F$  is essential for this relation to be useful for generating new solutions from given solutions. The usual continuum ( $M = \mathbb{R}^2$ ) Bäcklund transformation is obtained for  $F = \beta h^{-1}g$  where  $\beta$  is the spectral parameter (see [8], for example). For the nonlinear Toda lattice the relation

$$h_k^{-1}dh_k - g_k^{-1}dg_k = \beta *d(h_k^{-1}g_{k+1}) \quad (82)$$

reproduces the corresponding formulae in [4]. After some manipulation, for  $\tau = 0$  and  $\lambda \neq 0$  we find

$$\begin{aligned} g_k^{-1}\dot{g}_k &= \frac{1}{\lambda} \left( \beta h_{k-1}^{-1}g_k + \frac{1}{\beta} g_k^{-1}h_k + \alpha I \right) \\ h_k^{-1}\dot{h}_k &= \frac{1}{\lambda} \left( \beta h_k^{-1}g_{k+1} + \frac{1}{\beta} g_k^{-1}h_k + \alpha I \right) \end{aligned} \quad (83)$$

where  $\alpha$  is a constant of integration. Surprisingly, written in this way it generalizes to a Bäcklund transformation for the general chiral model on  $M = \mathbb{R} \times \mathbb{Z}$ . However, for a non-commutative group  $\mathcal{B}^*$  the above set of equations is no longer equivalent to (82), though there must be some  $F$  so that equation (81) is satisfied.

Now we turn to the remaining case  $M = \mathbb{Z}^2$ . Here we proceed in a different way. For an invertible matrix

$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (84)$$

with entries  $a, b, c, d \in \mathcal{B}$  and for  $z \in \mathcal{B}$  we define  $\mathcal{P}(\mathcal{M})z := (az + b)(cz + d)^{-1}$ . Using  $\mathcal{P}(\mathcal{M})\mathcal{P}(\mathcal{M}') = \mathcal{P}(\mathcal{M}\mathcal{M}')$  the field equations in the form (80) can now be rewritten as

$$\mathcal{P}(\hat{L}_k(n+1))\mathcal{P}(\hat{M}_k(n)) = \mathcal{P}(\hat{M}_{k+1}(n))\mathcal{P}(\hat{L}_k(n)). \quad (85)$$

For  $h_k(n) \in \mathcal{B}^*$  we define

$$h_{k+1}(n) = \mathcal{P}(\hat{L}_k(n))h_k(n) \quad h_k(n+1) = \mathcal{P}(\hat{M}_k(n))h_k(n). \quad (86)$$

The field equations for  $g_k(n)$  are now recovered as integrability conditions of this system, i.e. by calculating  $h_{k+1}(n+1)$  from both of the last two equations. The latter can be written in the form

$$\begin{aligned} h_{k+1}(n) &= c g_k(n) + \beta (c [g_k(n-1)]^{-1} - [h_k(n)]^{-1})^{-1} \\ c h_k(n+1) &= g_k(n) + \beta ([g_{k-1}(n)]^{-1} - c [h_k(n)]^{-1})^{-1} \end{aligned} \quad (87)$$

or, equivalently

$$\begin{aligned} g_{k+1}(n) &= c h_{k+1}(n+1) + \beta (c [h_{k+1}(n)]^{-1} - [g_k(n)]^{-1})^{-1} \\ c g_k(n+1) &= h_{k+1}(n+1) + \beta ([h_k(n+1)]^{-1} - c [g_k(n)]^{-1})^{-1}. \end{aligned} \quad (88)$$

The last set of equations shows that the field equations for  $g_k(n)$  are also obtained by calculating  $g_{k+1}(n+1)$  from both equations and comparing the results. As a consequence, if  $g_k(n)$  is a solution, then also  $h_k(n)$ . Hence (87) defines a Bäcklund transformation. Applied to the left movers (46) of the Toda model on  $\mathbb{Z}^2$  and writing  $h_k(n) = e^{-q_k(n)}$ , the transformation leads to  $q'_k(n+1) = c^{-2} q'_{k+1}(n)$  (together with a constraint on the initial values) and thus again to left movers.

## 7. Conclusions

We have presented a general method for integrable discretizations of two-dimensional chiral models via deformations of the ordinary differential calculus. As an example, the nonlinear Toda lattice is obtained in this way from the linear wave equation. It may be regarded as a  $GL(1, \mathbb{R})$  chiral model on  $\mathbb{R} \times \mathbb{Z}$ . More generally, our method works well in particular for  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  models. However, in the case of a chiral model for which the group algebra differs from the Lie algebra, the resulting discretized model in general suffers from unpleasant constraints. This has been demonstrated for the  $O(n)$   $\sigma$ -model. Perhaps there are modifications to our formalism which can improve such models.

The general formalism presented here also works for corresponding deformations of the differential calculus on  $\mathbb{R} \times S^1$  to calculi on a periodic (space) lattice. In that case, however, the first cohomology group is no longer trivial and (22) must no longer hold. But if the continuum chiral model is integrable, then so are its deformations. Furthermore, one may consider other (and in particular curved) metrics on the underlying two-dimensional space. In fact, the possibilities for generalizing the formalism developed in this work extend far beyond what we have mentioned up to this point. The deformed differential calculus of section 2 may be replaced by other differential calculi with a two-dimensional space of 1-forms (see [9] for candidates). All we need is a suitable generalization of the  $*$ -operator. It is then possible to generalize the (continuum) definition of a chiral model. This is what we have done in this work for a restricted class of differential calculi. It turned out that the nonlinear Toda lattice belongs to the corresponding extended class of chiral models. It remains to be seen whether other known integrable models can also be understood as generalized chiral models.

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